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A COMPUTER REALIZATION OF CONTROL-WITH-GUIDE PROCEDURES†

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Positional differential games of pursuit with target for-conflict-control systems, non-linear with respect to the phase vector, are considered. The problems investigated are approximate construction of the set of positional absorption and construction of control procedures guaranteeing guidance to the target. Issues relating to the development of algorithms for the approximate construction of the positional absorption set and control-with-guide procedures [1–4] are also examined. As an example to illustrate the possibilities of the algorithms considered, a pursuit game of the Homicidal Chauffeur type [5] is considered, with approximate computation of positional absorption sets in the problem of pursuit over a fixed time interval for several parameter sets. Motions of a conflict-control system that generate a control-with-guide procedure are computed for several specific initial values of the phase vector and several choices of the evader's control.

The present study is related to the research reported in [1–14].‡

1. FORMULATION OF THE PROBLEM

Consider a conflict-control system described in a time interval $[t_0, \vartheta]$ by a vector differential equation

$$dx/dt = F(t, x; u, v) \tag{1.1}$$
$$F(t, x; u, v) = f(t, x) + B(t, x)u + C(t, x)v;\ u \in P,\ v \in Q$$

where x is the phase vector of the system, which is a vector in the Euclidean space R^m, and u and v are the first and second players' control vectors, which satisfy the inclusions indicated in (1.1), where P and Q are convex compact polyhedra with a finite number of vertices in Euclidean spaces R^p and R^q, respectively.

It is assumed that the following conditions are satisfied

A. The functions $f(t, x)$, $B(t, x)$ and $C(t, x)$ are continuous jointly in their variables (t, x) in the domain $[t_0, \vartheta] \times R^m$, and for any bounded closed domain D ($D \subset [t_0, \vartheta] \times R^m$) Lipschitz constants $L\zeta = L(D)$ ($\zeta = f, B, C$) in $(0, \infty)$ exist such that

$$\|\zeta(t, x^{(1)}) - \zeta(t, x^{(2)})\| \leq L_\zeta \|x^{(1)} - x^{(2)}\|,\ \zeta = f, B, C \tag{1.2}$$

for any $(t, x^{(i)}) \in D$, $(i = 1, 2)$.

B. A constant $\tilde{\varkappa}$ in $(0, \infty)$ exists, such that, for all possible $(t, x, u, v) \in [t_0, \vartheta] \times R^m \times P \times Q$,

$$\|F(t, x; u, v)\| \leq \tilde{\varkappa}(1 + \|x\|) \tag{1.3}$$

The problem of pursuit facing the first player will be considered in two versions.

1. It is required to construct a positional mode of pursuit guaranteeing that the motion $x(t)$ of system (1.1) will reach, at time ϑ, a given compact target set M in R^m.

2. It is required to construct a positional mode of control guaranteeing that the motion $x(t)$ of system (1.1) will reach, in the time interval $[t_0, \vartheta]$, a given compact target set M in R^m.

†*Prikl. Mat. Mekh.* Vol. 66, No. 2, pp. 228–237, 2002.

‡See also: GUSYATNIKOV, P. B., A pursuit–evasion problem in the theory of differential games. Doctoral dissertation, Sverdlovsk, 1981

2. THE CONSTRUCTION OF THE POSITIONAL ABSORPTION SET

The positional modes of control considered below, which solve the pursuit problem, are positional control-with-guide procedures. The construction of controls in control-with-guide procedures makes essential use of positional absorption sets. It is well known that positional absorption sets may be defined on the basis of retrograde procedures [1].

Definition 1.1. A u-stable absorption operator $\pi(t_*; t^*, W^*)$ $(t_0 \leqslant t_* < t^* \leqslant \vartheta, W^* \subset R^m)$ is defined as the mapping $\pi(t_*; t^*, \cdot) : 2^{R^m} \to 2^{R^m}$ given by

$$\pi(t_*;t^*,W^*) = \bigcap_{v \in Q} X_v(t_*;t^*,W^*)$$

$$X_v(t_*;t^*,W^*) = \{x_* \in R^m : W^* \cap X_v(t^*;t_*,x_*) \neq \emptyset\}$$

where $X_v(t^*; t_*, x_*)$ is the set of all points in R^m which are reached at time t^* by solutions

$$x(\cdot) = (x(t)\colon t_* \leqslant t \leqslant t^*, x(t_*) = x_*)$$

of the differential inclusion

$$\dot{x} \in F_v(t,x), F_v(t,x) = F(t,x;P,v)$$

$$B(t,x)P = \{B(t,x)u\colon u \in P\}$$

The positional absorption set W^0 for Pursuit Problem 1 may be defined [1, 3] as the largest closed set $W \subset [t_0, \vartheta] \times R^m$ such that

$$W(\vartheta) \subset M, W(t_*) \subset \pi(t_*; t^*, W(t^*))$$

$$(W(t) = \{x \in R^m\colon (t,x) \in W\})$$

for any $t_*, t^* (t_0 \leqslant t_* < t^* \leqslant \vartheta)$.

By definition, W^0 is the set of all positions $(t_*, x_*) \in [t_0, \vartheta] \times R^m$ from which Pursuit Problem 1 is solvable. Exact computation of such subsets of the position space is possible for only the simplest classes of systems (1.1).

For the general case of systems (1.1), retrograde algorithms have been developed for approximate construction of the set W^0. These are discrete-time algorithms that compute a certain system of subsets of R^m approximating the set W^0.

The concept of an approximating system of sets arises when the continuous u-stability scheme is replaced by a discrete scheme. One introduces a partition $\Gamma = \{t_0, t_1, \ldots, t_N = \vartheta\}$ of the segment $[t_0, \vartheta]$ and replaces the reachable domains $X_v(t^*; t_*, x_*)$ by time-linear approximations.

Definition 1.2. An approximating u-stable absorption operator $\tilde{\pi}(t_*; t^*, W^*)$ $(t_0 \leqslant t_* < t^* \leqslant \vartheta, W^* \subset R^m)$ is a mapping $\tilde{\pi}(t_*; t^*, \cdot) : 2^{R^m} \to 2^{R^m}$ defined by

$$\tilde{\pi}(t_*;t^*,W^*) = \bigcap_{v \in Q} \tilde{X}_v(t_*;t^*,W^*)$$

$$\tilde{X}_v(t_*;t^*,W^*) = \{x_* \in R^m : W^* \cap \tilde{X}_v(t_*;t^*,x_*) \neq \emptyset\}$$

$$\tilde{X}_v(t^*;t_*,x_*) = x_* + (t^* - t_*)F_v(t_*,x_*)$$

where

$$(t^* - t_*)F_v(t_*,x_*) = \{a \in R^m : a = (t^* - t_*)b, b \in F_v(t_*,x_*)\}$$

An approximating system of sets (ASS) is defined on the basis of the concept of an approximating u-stable absorption operator. But before proceeding to the actual definition of an ASS, we will introduce some auxiliary quantities:

$$\omega_\zeta(\Delta) = \sup_{(t,x):(t,x)\in D,(t+\Delta,x)\in D} \|\zeta(t+\Delta,x)-\zeta(t,x)\|$$

$$K = \sup_{(t,x,u,v)\in D\times P\times Q} \|F(t,x;u,v)\|$$

$$K_1 = \max_{u\in P}\|u\|,\ K_2 = \max_{v\in Q}\|v\|,\ K, K_1, K_2 \in [0,\infty)$$

$$\omega^*(\Delta) = \omega_f(\Delta) + K_1\omega_B(\Delta) + K_2\omega_C(\Delta)$$

$$\lambda^* = L_f + K_1 L_B + K_2 L_C \in [0,\infty)$$

$$\Omega^*(\delta, r) = \omega^*(\delta) + \lambda^* r, \quad \delta \geqslant 0, r \geqslant 0$$

$$\lambda = K\lambda^*,\ \varkappa(\Delta) = \omega^*(\Delta) + \lambda\Delta\ (\Delta \geqslant 0)$$

Letting $\rho(F_*, F^*)$ denote the Hausdorff distance between F_* and F^*, we conclude that in the problem under consideration the following condition holds

$$\rho(F_v(t,x^*), F_v(t,x_*)) \leqslant \lambda^* \|x^* - x_*\|$$

for any (t,x_*), (t,x^*) in D, $v \in Q$, and moreover

$$\rho(F_v(t^*,x^*), F_v(t_*,x_*)) \leqslant \Omega^*(|t^* - t_*|, \|x^* - x_*\|)$$

for any (t_*,x_*), (t^*,x^*) in D, $v \in Q$. Therefore, by the scheme of [4], one obtains a rigorous definition of an ASS:

Definition 1.3. An approximating system of sets $\{\tilde{W}(t_i) : t_i \in \Gamma\}$ is defined as a system for which

$$\tilde{W}(t_N) = M_{\varepsilon_N},\ \tilde{W}(t_i) = \tilde{\pi}(t_i;\ t_{i+1}, \tilde{W}(t_{i+1})),\ i = N-1,\dots,0$$

where the number ε_N is found from the recurrence relations

$$\varepsilon_0 = 0,\ \varepsilon_i = \omega(\Delta_{i-1}) + (1+\lambda^*\Delta_{i-1})\varepsilon_{i-1},\ i = 1,\dots,N-1$$

$$\Delta_i = t_{i+1} - t_i, t_i \in \Gamma; \omega(\Delta) = \Delta\Omega^*(\Delta, K\Delta) = \Delta\omega^*(\Delta) + \lambda^* K\Delta^2$$

The symbol Φ_ε denotes a closed ε-neighbourhood of the set Φ.

By Theorem 1 of [4], an ASS $\{\tilde{W}(t_i) : t_i \in \Gamma\}$ converges as the diameter of the partition $\Delta(\Gamma)$ approaches zero to a positional absorption set W^0. Consequently, the set W^0 can be computed approximately as a system of sets $\{\tilde{W}(t_i) : t_i \in \Gamma\}$. In addition, considering some sufficiently fine mesh $S^{(u)} = \{u_\gamma: u_\gamma \in \partial P, \gamma = 1,\dots,k\}$ such that $\mathrm{co}S^{(u)} = P$, and also $S^{(v)} = \{u_\omega: u_\omega \in \partial Q, \omega = 1,\dots,n\}$, of vertices of the polyhedron Q, we conclude [4] that the sets $\tilde{W}(t_i)$ $(i = N,\dots,0)$ can be computed approximated by the formulae

$$\tilde{W}(t_N) = \tilde{M},\ \tilde{W}(t_i) = \bigcap_{\omega=1}^{n} \bigcup_{\gamma=1}^{k} \tilde{X}_{\omega,\gamma}(t_i; t_{i+1}, \tilde{W}(t_{i+1})) \tag{2.1}$$

$$i = N-1,\dots,0$$

where

$$\tilde{X}_{\omega,\gamma}(t_i; t_{i+1}, \tilde{W}(t_{i+1})) = \{x[t_i] : \tilde{W}(t_{i+1}) \cap \tilde{X}_{\omega,\gamma}(t_{i+1}; t_i, x[t_i]) \neq \emptyset\}$$

$$\tilde{X}_{\omega,\gamma}(t_{i+1}; t_i, x[t_i]) = x[t_i] + \Delta_i F(t_i, x[t_i]; u_\gamma, v_\omega)$$

$\tilde{M}$ is a polyhedron in R^m with a finite number of vertices, approximating the target set M.

It is known that if $\tilde{W}(t_{i+1})$ is a polyhedron with a finite number of vertices, then the set $\tilde{X}_{\omega,\gamma}(t_i; t_{i+1}, \tilde{W}(t_{i+1}))$ will be, to a known degree of accuracy, a polyhedron in R^m with the same number of vertices. The vertices of the polyhedron $\tilde{X}_{\omega,\gamma}(t_i; t_{i+1}, \tilde{W}(t_{i+1}))$ are computed, given the vertices $x[t_{i+1}]$ of $\tilde{W}(t_{i+1})$, from the equation

$$x + \Delta_i F(t_i, x; u_\gamma, v_\omega) = x[t_{i+1}]$$

The solution of this equation may be found by successive approximations, to any desired accuracy, provided that the numbers Δ_i are sufficiently small. In some cases, such as that of the Homicidal Chauffeur model problem, an exact solution can be found.

Thus, the sets $\tilde{W}(t_i)$ occurring in (2.1) are finite intersections of finite unions of not necessarily convex polyhedra with a finite number of vertices in the space R^m. Unions and intersections of such polyhedra in the space R^2 have been computed for various specific problems on the basis of existing algorithms [14].

An approximate construction of positional absorption sets by approximating systems, analogous to the construction described above for Pursuit Problem 1, has also been worked out for Pursuit Problem 2 [4].† The basic element of the approximate computation of positional absorption sets in this construction is the same procedure as is used to compute unions and intersections of polyhedra with a finite number of vertices in Euclidean space.

Thus, when the set M can be represented as a union of spheres of radii bounded below by some positive number R^*, a system of sets approximating the positional absorption set in Pursuit Problem 2 may be given by the relations

$$\tilde{W}(t_N) = \tilde{M}_{\varepsilon_N}, \tilde{W}(t_i) = \tilde{\pi}(t_i; t_{i+1}, \tilde{W}(t_{i+1}) \cup M_{\varepsilon_{i+1}})$$

$$i = N-1, N-2, \ldots, 0,$$

where $\{\varepsilon_i\}$ is some sequence of numbers such that $\varepsilon_0 = 0$ and $\max_{1\le i\le N} \varepsilon_i \to 0$ as $\Delta(\Gamma) \to 0$.

Just as in the case of Pursuit Problem 1, the sets $\tilde{W}(t_i)$ $(i = 0, \ldots, N)$ can be computed approximately from the formulae

$$\tilde{W}(t_N) = \tilde{M}, \quad \tilde{W}(t_i) = \bigcap_{\omega=1}^{n} \bigcup_{\gamma=1}^{k} \tilde{X}_{\omega,\gamma}(t_i; t_{i+1}, \tilde{W}(t_{i+1}) \cup \tilde{M})$$

$$i = N-1, N-2, \ldots, 0$$

$$\tilde{X}_{\omega,\gamma}(t_i; t_{i+1}, \tilde{W}(t_{i+1})) = \{x[t_i] \in R^m : (\tilde{W}(t_{i+1}) \cup \tilde{M}) \cap \tilde{X}_{\omega,\gamma}(t_{i+1}; t_i, x[t_i]) \ne \emptyset\}$$

where $\tilde{M}$ is a polyhedron approximating the target set M.

As in the case of Pursuit Problem 1, a procedure to compute the sets $\tilde{W}(t_i)$ $(i = 0, 1, \ldots, N)$ can be realized for specific Pursuit Problems 2 in the plane R^2. For example, we shall consider below a pursuit problem of the Homicidal Chauffeur type. The solution of this problem can be approached by solving Pursuit Problem 2 for a suitable conflict-control system over different time segments $[t_0, T]$. The pursuit problem itself may be solved over a fixed time segment $[t_0, T]$, at least approximately, using the approximating constructions proposed above for Pursuit Problem 2.

3. CONTROL PROCEDURES

A convenient positional mode of control for solving pursuit problems is the positional control-with-guide procedure [1, 2]. One positive property of these control procedures is that the motions they generate are stable under small perturbations of the phase vector of the system. Another positive property is their simple computer execution, at least, for systems of type (1.1) in a plane. The most difficult element in the execution of control-with-guide procedures, namely, the computation of the intersection of a bundle of motions with a section of a bridge, may be executed in the two-dimensional case using the aforementioned procedures for computing intersections of polyhedra (see [14]).

†See also: TARAS'YEV, A. M. and USHAKOV, V. N., The construction of stable bridges in a minimax game of pursuit and evasion. Sverdlovsk, 1983. Dep. at the All-Union Institute for Scientific and Technical Information (VINITI) 5.05.83, No. 2454–83.

In this section we shall consider positional control-with-guide procedures in which suitable controls in actual motion and a guide are chosen on the basis of the idea of copying [6, 7]. When that is done, the second player's control in the guide is chosen on the basis of copying with delay.

Considering Pursuit Problem 1, we shall construct a "solving mode of control" as a positional control-with-guide procedure for the first player. We may assume without loss of generality that the motions generated by the procedure which reach an ε-neighbourhood of the target M must remain in some known bounded domain D. This domain D will occur in the subsequent reasoning.

We shall agree in advance that the auxiliary motion of the guide is constructed as a motion that passes through the system of sets approximating the positional absorption set. Accordingly, the solving control procedure of which we shall be speaking below will steer the actual motion of system (1.1) not exactly to the target M but to an ε-neighbourhood of the target. The magnitude of ε will depend on the diameter $\Delta(\Gamma)$ of the partition Γ relative to which the ASS is considered, and $\varepsilon \to 0$ as $\Delta(\Gamma) \to 0$.

Thus, assuming that an ASS $\{\tilde{W}(t_i) : t_i \in \Gamma\}$ is given [4], let us define a positional control-with-guide procedure for the first player corresponding to the partition Γ.

We introduce the following notation

$$Y_k(t) = (t - t_k)F(t_k, y[t_k]; u_*^k, v_*^k)$$

$$X_k(t) = \int_{t_k}^{t} F(\tau, x[\tau]; u^k, v(\tau))d\tau$$

$$H^k = \Delta F(t_k, x[t_k]; u^k, v^k) + h^k$$

$$h^k = \int_{t_k}^{t_{k+1}} (F(\tau, x[\tau]; u^k, v^k) - F(t_k, x[t_k]; u^k, v^k))d\tau$$

$$v^k = \frac{1}{\Delta}\int_{t_k}^{t_{k+1}} v(\tau)d\tau$$

$$z_k = \|y[t_k] - x[t_k]\|, \ k = 0, \ldots, N-1$$

$$\varphi(\Delta) = K_2(\vartheta - t_0)\omega_C(\Delta) + L_C K K_2(\vartheta - t_0)\Delta + 2(\vartheta - t_0)\Delta\varkappa(\Delta) +$$

$$+\Delta \| C(t_0, x[t_0])v_*^0 \| + \Delta K K_2 + (\vartheta - t_0)\varkappa(\Delta)$$

Let $x[t_0]$ be some point in R^m. We wish to find the point in the set $\tilde{W}(t_0)$, say $y[t_0]$, that is closest to $x[t_0]$.

In the interval $[t_0, t_1]$ we assume that a control for the second player in the guide is given as a vector $v_*^0 \in Q$, and a control for the second player as a vector $u_*^0 \in P$ is then found from the condition $y[t_1] \in \tilde{W}(t_1)$, where $y[t] = y[t_0] + Y_0[t]$.

We shall refer to the vector function $y[t_0]$ in $[t_0, t_1]$ as the motion of the guide.

The first player's control in the interval $[t_0, t_1)$ in the actual system is a vector $u^0 \in P$ defined by $u^0 = u_*^0$. Let us assume that in the interval $[t_0, t_1)$ in the actual system some measurable control $v[t] \in Q$, $t \in [t_0, t_1)$ is achieved. Then the motion of the actual system in $[t_0, t_1]$ satisfies the equality

$$x[t] = x[t_0] + X_0[t]$$

Obviously

$$x[t_1] = x[t_0] + H^0$$

We have the inequality $\|h^0\| \leqslant \Delta\varkappa(\Delta)$.

Now, proceeding by induction, let us assume that in the interval $[t_{k-1}, t_k)$ $(1 \leqslant k \leqslant N-1)$ we have given controls u_*^{k-1}, v_*^{k-1}, u^{k-1}, $v^{k-1}(\cdot)$, as well as the motions of the guide and of the actual system $y[\cdot](y[t_k] \in \tilde{W}(t_k))$, $x[\cdot]$ in $[t_{k-1}, t_k]$ generated by these controls. We define the controls, motions of the guide and of the actual system $y[\cdot], x[\cdot]$ in the segment $[t_k, t_{k+1}]$ as follows.

The vector $v_*^k \in Q$ is found from the condition

$$\left\| x[t_k] - x[t_{k-1}] - \Delta F(t_{k-1}, x[t_{k-1}]; u^{k-1}, v_*^k) \right\| \leqslant \Delta\varkappa(\Delta)$$

We have the inequality

$$\left\| \Delta C(t_{k-1}, x[t_{k-1}])\, v_*^k - \Delta C(t_{k-1}, x[t_{k-1}])\, v^{k-1} \right\| \leq 2\Delta\varkappa(\Delta)$$

We now define the vector $u_*^k \in P$ by requiring that $y[t_{k+1}] \in \tilde{W}(t_{k+1})$, where $y[t] = y[t_k] + Y_k(t)$, $t \in [t_k, t_{k+1}]$. Such a vector u_*^k exists, since $y[t_k]$ satisfies the inclusion $y[t_k] \in \tilde{W}(t_k)$.

The vector function $y[t_k]$ in $[t_k, t_{k+1}]$ will be called the motion of the guide.

The vector $u^k \in P$ is defined by the equality $u^k = u_*^k$.

We define the motion of the actual system in $[t_k, t_{k+1}]$ by the relation

$$x[t] = x[t_k] + X_k(t),$$

where $v(t)$, $t \in [t_k, t_{k+1})$ is the second player's control thus realized.

We have the following representation for the point $x[t_{k+1}]$

$$x[t_{k+1}] = x[t_k] + H^k$$

We have the inequality $\|h^k\| \leq \Delta\varkappa(\Delta)$.

Thus, running through all the numbers $k = 0, 1, \ldots, N-1$ in succession, we define a positional control-with-guide procedure for the first player, corresponding to the partition Γ.

We will now estimate the quantity z_{k+1} at the nodal points $t_{k+1} \in \Gamma$ ($k = 0, 1, \ldots, N-1$) in terms of the initial deviation z_0.

The following equalities hold

$$y[t_{k+1}] - x[t_{k+1}] = (y[t_k] - x[t_k]) + \Delta(F(t_k, y[t_k]; u^k, v^k) - F(t_k, x[t_k]; u^k, v^k)) - h^k$$

$$k = 0, 1, \ldots, N-1$$

Hence we obtain

$$z_{k+1} \leq z_0 + \Delta L_f \sum_{i=0}^{k} z_i + \Delta L_B K_1 \sum_{i=0}^{k} z_i + \Delta L_C K_2 \sum_{i=0}^{k} z_i + \Delta k\omega_C(\Delta) K_2 + \Delta k L_C K \Delta K_2 +$$

$$+ \Delta k 2\Delta\varkappa(\Delta) + \Delta \left\| C(t_0, x[t_0])\, v_*^0 \right\| + \Delta K K_2 + k\Delta\varkappa(\Delta)$$

$$k = 0, 1, \ldots, N-1$$

Taking the inequalities $k\Delta \leq \vartheta - t_0$ into account, we obtain an estimate

$$z_{k+1} \leq z_0 + \lambda\Delta \sum_{i=0}^{k} z_i + \varphi(\Delta)$$

$$k = 0, 1, \ldots, N-1$$

Then

$$z_1 \leq z_0 + \lambda\Delta z_0 + \varphi(\Delta) \leq e^{\lambda\Delta} z_0 + \varphi(\Delta)$$

$$z_2 \leq z_0 + \lambda\Delta(z_0 + z_1) + \varphi(\Delta) \leq e^{2\lambda\Delta} z_0 + e^{\lambda\Delta}\varphi(\Delta)$$

Taking these inequalities into account, one can prove that

$$z_k \leq e^{\lambda(\vartheta - t_0)} z_0 + e^{\lambda(\vartheta - t_0)}\varphi(\Delta), \quad k = 0, 1, \ldots, N$$

4. EXAMPLE

Let us consider the Homicidal Chauffeur differential game [5]. The pursuer is moving in a plane at a fixed velocity w_1, in a trajectory whose radius of curvature is bounded below by a positive number R. The pursuer controls the choice of the actual value of the curvature of the trajectory at each instant of time. The evader, moving in the same plane, executes a simple motion, with fixed velocity w_2. The pursuer tries to entrap the evader in a prescribed l-neighbourhood as rapidly as possible. The equations of motion in this pursuit game in the reduced space have the form

$$\begin{aligned}\dot{x}_1 &= -w_1R^{-1}x_2\varphi + w_2\sin\psi\\ \dot{x}_2 &= w_1R^{-1}x_1\varphi + w_2\cos\psi - w_1\end{aligned} \tag{4.1}$$

The players' controls φ and ψ satisfy the conditions

$$-1 \leqslant \varphi \leqslant 1, 0 \leqslant \psi \leqslant 2\pi$$

Closely connected with this pursuit problem is the problem of the system approaching the target M in a fixed time segment $[0, T]$, $T \in (0, \infty)$. To be precise: the isochrones in the pursuit problem are parts of the boundary of the corresponding sections of the positional absorption set, and a solving strategy in the pursuit problem can be constructed as a control-with-guide procedure or as a strategy extremal to the positional absorption set.

System (3.1) can be replaced by the following system

$$\begin{aligned}&\dot{x}_1 = -w_1R^{-1}x_2u_1 + w_2v_1\\ &\dot{x}_2 = w_1R^{-1}x_1u_2 + w_2v_2 - w_1\\ &u = (u_1, u_2) \in P, v = (v_1, v_2) \in Q\\ &P = \{u : u_1 = u_2, -1 \leqslant u_1 \leqslant 1\}, Q = \{v : \|v\| \leqslant 1\}\end{aligned} \tag{4.2}$$

Equations (4.2) are non-linear differential equations of the form (1.1), and the algorithms considered in the previous sections are applicable. As indicated previously, the algorithms consist of the following three main parts.

1. Approximation of the sets P, Q and M by polyhedra.
2. Solution of the non-linear equation

$$x + (t_{i+1} - t_i)F(t_i, x; u_\gamma, v_\omega) = x_s[t_{i+1}] \tag{4.3}$$

in each segment $[t_i, t_{i+1}]$ of the partition Γ for different sets of vectors u_γ, v_ω and $x_s[t_{i+1}]$ – the vertices of the corresponding polyhedra approximating the sets P, Q and M.

3. Construction of intersections and unions of non-convex non-simply connected polyhedra in R^2 (see [14]).

To construct polyhedra approximating the sets P, Q and M, one defines meshes on their boundaries. The choice of the number of elements in these approximating polyhedra depends on the computational resources and on the required precision.

Incidentally, in this particular game the solution $(x_1[t_i], x_2[t_i])$ of Eq. (4.3) can be written out analytically. In the general case, however, one has to solve such equations approximately.

The system of sets $\{\tilde{W}(t_i) : t_i \in \Gamma\}$ was constructed for the following parameter values

$$w_1 = 2.5, w_2 = 1, R = 0.5, l = 1, [t_0, \vartheta] = [0, 2]$$

$$\Gamma = \{t_0 = 0, t_1, \ldots, t_N = 2\}, \Delta = t_{i+1} - t_i = 0.05, N = 100$$

The sets P, Q and M are approximated by polyhedra in the plane with suitable vertices

$$u_\gamma = (u_{\gamma 1}, u_{\gamma 2}):\ u_{\gamma 1} = u_{\gamma 2}, u_{\gamma 1} = \left(1 - \frac{\gamma}{2}\right)/\cos\frac{\pi}{4}$$

$$\gamma = 0, \ldots, S_p, S_p = 4$$

$$v_\omega = (v_{\omega 1}, v_{\omega 2}):\ v_{\omega 1} = \cos\frac{2\pi\omega}{S_q},\ \ v_{\omega 2} = \sin\frac{2\pi\omega}{S_q}$$

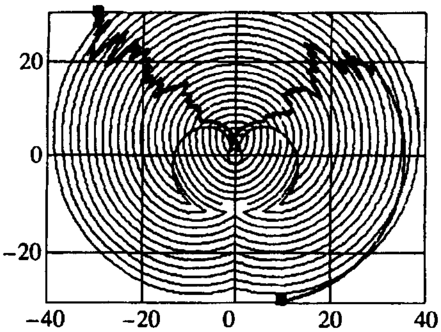

Fig. 1

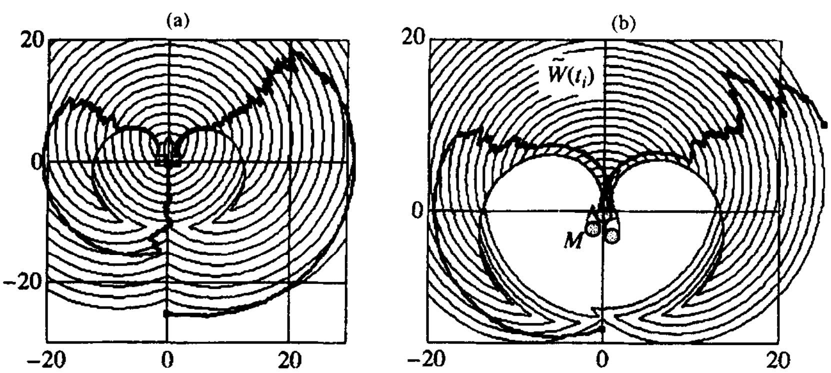

Fig. 2

$$\omega = 1, 2, \ldots, S_q, S_q = 8$$

$$x_s = (x_{s1}, x_{s2}): \; x_{s1} = l\cos\frac{2\pi s}{S_m}, \; x_{s2} = l\sin\frac{2\pi s}{S_m}$$

$$s = 1, 2, \ldots, S_m, S_m = 60$$

Figure 1 represents the sets $\tilde{W}(t_i): t_i \in \Gamma$ in the motion of system (4.2) for two different initial positions, as indicated by markers. The evader chose the control in a random manner. Figure 2 illustrates the motions of system (4.2) if the target set M is given by two squares (a) or three disks (b).

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